APPENDIX A
DERIVATION OF THE ERROR BOUND

To reduce the error $\Delta\hat{\mathcal{I}}$, our method aims to select the triplet with the maximum standard deviation. The standard deviation σ_j for the j-th triplet is calculated from (the arguments are omitted for simplicity):

$$\sigma_j = \sqrt{\mathbb{V}ar\,[\hat{\mathcal{I}}_j]} = \frac{A_j W_j I_j}{\sqrt{n_j}} \sqrt{\mathbb{V}ar\,[R_d H]}.$$

Since the computation of the variance of $R_d H$ involves costly evaluations of $R_d H$ over all VPLs, irradiance points, and shading points in the triplet $(\mathbb{C}_j^L, \mathbb{C}_j^I, \mathbb{C}_j^S)$, and the standard deviation is just used to select the triplet with (possibly) a large error, our method approximates the standard deviation using the upper bounds of the diffuse BSSRDF R_d^u, the geometry term G^u, and the Fresnel transmittance F_t^u as follows:

$$\sigma_j = \frac{A_j W_j I_j}{\sqrt{n_j}} \sqrt{\mathbb{V}ar\,[R_d G V F_t]} \approx \frac{A_j W_j I_j}{\sqrt{n_j}} R_d^u G^u F_t^u \sqrt{\mathbb{V}ar\,[V]}.$$

The standard deviation of the visibility function, $\sqrt{\mathbb{V}ar\,[V]}$, is bounded by the maximum value 0.5 [1]. By substituting this, the error bound e_j is derived.

A.1 The geometry term G and the upper bounds G^u and F_t^u

The geometry term $G(\mathbf{y}, \mathbf{x}_i)$ between the VPL $\mathbf{y}$ and the irradiance point $\mathbf{x}_i$ is calculated as:

$$G(\mathbf{y}, \mathbf{x}_i) = \frac{\max(\cos\theta_{\mathbf{y}}, 0)\max(\cos\theta_{\mathbf{x}_i}, 0)}{\|\mathbf{y} - \mathbf{x}_i\|^2},$$

where $\theta_{\mathbf{y}}$ is the angle between the normal at $\mathbf{y}$ and $-\omega_{\mathbf{y}\mathbf{x}_i}$, and $\theta_{\mathbf{x}_i}$ is that between the normal $\mathbf{n}_i$ at $\mathbf{x}_i$ and $\omega_{\mathbf{y}\mathbf{x}_i}$.

The upper bound G^u of the geometry term between IC $\mathbb{C}_j^I$ and LC $\mathbb{C}_j^L$ is calculated in the same way as Multidimensional Lightcuts [2]. Since the computation of the upper bound G^u involves computation of the upper bound of $\cos\theta_{\mathbf{x}_i}$ (i.e., the lower bound of $\theta_{\mathbf{x}_i}$), and the Fresnel transmittance F_t decreases monotonically [1]with respect to the angle $\theta_{\mathbf{x}_i}$, the upper bound F_t^u between $\mathbb{C}_j^I$ and $\mathbb{C}_j^L$ is easily computed using the lower bound of $\theta_{\mathbf{x}_i}$.

A.2 The diffuse BSSRDF $R_d(r)$ and its upper bound R_d^u

The diffuse BSSRDF $R_d(r)$ is represented by the following equation [6]:

$$\frac{\alpha'}{4\pi}\left[z_r\left((1+\sigma_{tr}d_r)\frac{e^{-\sigma_{tr}d_r}}{d_r^3}\right) + z_v\left((1+\sigma_{tr}d_v)\frac{e^{-\sigma_{tr}d_v}}{d_v^3}\right)\right],$$

$$d_r = \sqrt{r^2 + z_r^2}, d_v = \sqrt{r^2 + z_v^2}, z_r = \frac{1}{\sigma_t'}, z_v = z_r\left(1 + \frac{4}{3}\mathcal{A}\right),$$

$$\sigma_{tr} = \sqrt{3\sigma_a\sigma_t'}, \sigma_t' = \sigma_s' + \sigma_a, \sigma_s' = \sigma_s(1-g), \alpha' = \frac{\sigma_s'}{\sigma_t'}$$

where σ_a, σ_s, and σ_t are absorption, scattering, and extinction coefficients for a translucent material, σ_s' and σ_t' are reduced scattering and extinction coefficients, g is the mean cosine of the scattering angle, σ_{tr} is the effective extinction coefficient, $\mathcal{A}$ is calculated by the diffuse Fresnel reflectance.

The diffuse BSSRDF $R_d(r)$ is also a monotonically decreasing function with respect to the distance r between an irradiance point and a shading point. As such, the upper bound R_d^u is calculated from the minimum distance between the bounding boxes of IC $\mathbb{C}_j^I$ and SC $\mathbb{C}_j^S$.

1. Strictly speaking, F_t decreases monotonically when the relative refractive index η satisfies $\eta \in [2-\sqrt{3}, 2+\sqrt{3}]$ [3], and η for translucent materials used in our method satisfies this condition.

REFERENCES

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