Supplemental Document : Resampling-aware Weighting Functions for Bidirectional Path Tracing using Multiple Light Sub-Paths

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This supplemental document includes the derivation of the variance for our *resampling estimator* (Sec. 1 and Sec. 2) and our *resampling-aware weighting functions* (Sec. 3). The derivation of the conditional variance $V[\hat{I}_t|\bar{Z}_t]$ (Eq. (8) in the paper) is shown in Sec. 1, where $\hat{I}_t$ is the resampling estimator and $\bar{Z}_t$ is the eye sub-path sample with t vertices. The derivation of the variance $V[\hat{I}_t]$ of the resampling estimator $\hat{I}_t$ (Eqs. (9) and (10) in the paper) is shown in Sec. 2. This supplemental document is also intended to show additional results (San Miguel scene) and details of Fig. 4 in the main paper.

1 DERIVATION OF THE VARIANCE $V[\hat{I}_t|\bar{Z}_t]$

The resampling estimator $\hat{I}_t$ estimates the contributions from all the paths having eye sub-path with t vertices. The resampling estimator estimates the following integral I_t as:

$$I_t = \int_{\Omega_t} w_t(\bar{x}) f(\bar{x}) d\mu(\bar{x}),$$

where Ω_t is the space of the paths whose eye sub-paths have t vertices, and w_t is the weighting function for the resampling estimator. We define $g(\bar{x}) = w_t(\bar{x}) f(\bar{x})$ to simplify the notation. To compute I_t, we first sample an eye sub-path with t vertices as:

$$I_t \approx \frac{1}{p(\bar{Z}_t)} \int_{\mathcal{A}} g(\bar{y}\bar{Z}_t) d\mu(\bar{y}) = \frac{1}{p(\bar{Z}_t)} \sum_{s \geq 1} \int_{A^s} g(\bar{y}\bar{Z}_t) d\mu(\bar{y}),$$

where $\bar{Z}_t$ is the eye sub-path sample with t vertices, p is the sampling pdf, $\bar{y}$ is an integral variable of light sub-paths, and $\bar{y}\bar{Z}_t$ is a full light path by connecting the last vertex of the light sub-path $\bar{y}$ and the t-th vertex

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Table 1. Notations and symbols. Subscripts i and j represent the index of the sample. Subscripts s and t indicate the number of vertices for light and eye sub-paths, respectively.

symbol	meaning
f	measurement contribution function
$\bar{x}$	full light path $x_0 \dots x_k$
$\bar{y}$	light sub-path
$\bar{z}$	eye sub-path
$\bar{Y}_{s,i}$	i-th light sub-path sample with s vertices
$\bar{Z}_t$	eye sub-path sample with t vertices
$\bar{\mathbf{Y}}$	a set of proposals $\{\bar{Y}_{1,1}, \dots, \bar{Y}_{1,M}, \dots, \bar{Y}_{s,i}, \dots\}$
$\bar{X}_{s,i}$	full light path sample connecting $\bar{Y}_{s,i}$ and $\bar{Z}_t$
$\bar{\mathbf{X}}$	a set of proposals $\{\bar{X}_{1,1}, \dots, \bar{X}_{1,M}, \dots, \bar{X}_{s,i}, \dots\}$
$\bar{X}_j$	j-th sample from $\bar{\mathbf{X}}$
M	number of light sub-path samples
N	number of resampling light sub-paths per iteration
$A, A^s, \mathcal{A}$	scene suface, s-dimensional product of A, union of A^s
$q^*(\bar{x})$	target distribution (a part of contribution function f)
$q(\bar{x}), q_s(\bar{x})$	target pdfs (normalize q^* over $\mathcal{A}$ and A^s)
Q, Q_s	normalization factors (integrate q^* over $\mathcal{A}$ and A^s)

of the eye sub-path sample $\bar{Z}_t$. $\mathcal{A}$ is the integral domain, which is defined as $\mathcal{A} = \cup_{s \geq 1} A^s$, where A^s is the s-dimensional Cartesian product over the scene surface A.

Our method estimates the integral I_t by using RIS with partitioned integral domain A^s. For each integral domain A^s, M light sub-path samples $\{\bar{Y}_{s,1}, \dots, \bar{Y}_{s,M}\}$ are prepared to estimate each integral over A^s.

$$\frac{1}{p(\bar{Z}_t)} \sum_{s \geq 1} \int_{A^s} g(\bar{y}\bar{Z}_t) d\mu(\bar{y}) = \frac{1}{p(\bar{Z}_t)} \sum_{s \geq 1} \frac{1}{M} \sum_{i=1}^{M} \frac{g(\bar{Y}_{s,i}\bar{Z}_t)}{p(\bar{Y}_{s,i})}, \tag{1}$$

where $\bar{Y}_{s,i}\bar{Z}_t$ is a full light path connecting the light sub-path sample $\bar{Y}_{s,i}$ and the eye sub-path sample $\bar{Z}_t$.

By drawing N samples from the set of proposals $\bar{\mathbf{Y}} = \{\bar{Y}_{1,1}, \dots, \bar{Y}_{1,M}, \dots, \bar{Y}_{s,1}, \dots, \bar{Y}_{s,M}, \dots\}$, the resampling estimator $\hat{I}_t$ is given by:

$$\hat{I}_t = \frac{1}{MN} \sum_{j=1}^{N} \frac{g(\bar{Y}_j \bar{Z}_t)}{p(\bar{Y}_j) p(\bar{Z}_t) P_r(\bar{Y}_j | \bar{\mathbf{Y}})},$$

$$P_r(\bar{Y}_j | \bar{\mathbf{Y}}) = \frac{q^*(\bar{Y}_j \bar{Z}_t)/p(\bar{Y}_j)}{\sum_{s \geq 1} \sum_{i=1}^{M} q^*(\bar{Y}_{s,i}\bar{Z}_t)/p(\bar{Y}_{s,i})},$$

where $\bar{Y}_j$ is the j-th sample from the set of proposals $\bar{\mathbf{Y}}$ and P_r is referred to as the resampling pmf.

To simplify the notation, we represent a full light path sample $\bar{Y}_{s,i}\bar{Z}_t$ and $\bar{Y}_j\bar{Z}_t$ with $\bar{X}_{s,i}$ and $\bar{X}_j$ respectively. We also represent the set of proposals as $\bar{\mathbf{X}} = \{\bar{X}_{1,1}, \dots, \bar{X}_{1,M}, \dots, \bar{X}_{s,1}, \dots, \bar{X}_{s,M}, \dots\}$. By using these notations, the resampling pmf $P_r(\bar{X}_j | \bar{\mathbf{X}})$, which is equivalent to $P_r(\bar{Y}_j | \bar{\mathbf{Y}})$, is expressed by:

$$P_r(\bar{X}_j | \bar{\mathbf{X}}) = \frac{q^*(\bar{X}_j)/p(\bar{X}_j)}{\sum_{s \geq 1} \sum_{i=1}^{M} q^*(\bar{X}_{s,i})/p(\bar{X}_{s,i})} = \frac{q^*(\bar{Y}_j \bar{Z}_t)/p(\bar{Y}_j)\cancel{p(\bar{Z}_t)}}{\sum_{s \geq 1} \sum_{i=1}^{M} q^*(\bar{Y}_{s,i}\bar{Z}_t)/p(\bar{Y}_{s,i})\cancel{p(\bar{Z}_t)}} = P_r(\bar{Y}_j | \bar{\mathbf{Y}}), \tag{2}$$

where $p(\bar{X}_j) = p(\bar{Y}_j)p(\bar{Z}_t)$ and $p(\bar{X}_{s,i}) = p(\bar{Y}_{s,i})p(\bar{Z}_t)$ are used. By using the resampling pmf $P_r(\bar{X}_j|\bar{\mathbf{X}})$, the resampling estimator $\hat{I}_t$ is rewritten as:

$$\hat{I}_t = \frac{1}{MN}\sum_{j=1}^{N}\frac{g(\bar{Y}_j\bar{Z}_t)}{p(\bar{Y}_j)p(\bar{Z}_t)P_r(\bar{Y}_j|\bar{\mathbf{Y}})} = \frac{1}{MN}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{p(\bar{X}_j)P_r(\bar{X}_j|\bar{\mathbf{X}})}. \tag{3}$$

The conditional variance $V[\hat{I}_t|\bar{Z}_t]$ for the eye sub-path $\bar{Z}_t$ is expressed as:

$$V[\hat{I}_t|\bar{Z}_t] = E[\hat{I}_t^2|\bar{Z}_t] - E[\hat{I}_t|\bar{Z}_t]^2. \tag{4}$$

We first describe the derivation of the conditional expected value $E[\hat{I}_t|\bar{Z}_t]$ in Sec. 1.1, then that of $E[\hat{I}_t^2|\bar{Z}_t]$ is described in Sec. 1.2.

1.1 Derivation of $E[\hat{I}_t|\bar{Z}_t]$

The conditional expected value $E[\hat{I}_t|\bar{Z}_t]$ is calculated by the following equation:

$$E[\hat{I}_t|\bar{Z}_t] = E\left[\frac{1}{MN}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{p(\bar{X}_j)P_r(\bar{X}_j|\bar{\mathbf{X}})}|\bar{Z}_t\right]. \tag{5}$$

By substituting Eq. (2) into Eq. (5), $E[\hat{I}_t|\bar{Z}_t]$ is given by:

$$E[\hat{I}_t|\bar{Z}_t] = E\left[\frac{1}{MN}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{p(\bar{X}_j)}\frac{\sum_{s\geq 1}\sum_{i=1}^{M}q^*(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{q^*(\bar{X}_j)/p(\bar{X}_j)}|\bar{Z}_t\right] = E\left[\left(\frac{1}{N}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{q^*(\bar{X}_j)}\right)\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right]. \tag{6}$$

By using $E[ab] = E[E[a|b]b]$, the expected value of the product $\frac{1}{N}\sum_{j=1}^{N} g/q^*$ and $\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M} q^*/p$ is calculated by:

$$E\left[\hat{I}_t|\bar{Z}_t\right] = E\left[E\left[\frac{1}{N}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{q^*(\bar{X}_j)}|\bar{\mathbf{X}}\right]\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right]. \tag{7}$$

Since the samples $\bar{X}_j$ are independent and identically distributed, the expected values of the samples are identical as:

$$E\left[\frac{1}{N}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{q^*(\bar{X}_j)}|\bar{\mathbf{X}}\right] = \frac{1}{N}\sum_{j=1}^{N}E\left[\frac{g(\bar{X}_1)}{q^*(\bar{X}_1)}|\bar{\mathbf{X}}\right] = E\left[\frac{g(\bar{X}_1)}{q^*(\bar{X}_1)}|\bar{\mathbf{X}}\right]. \tag{8}$$

In addition, since $\bar{X}_1$ is one of the set $\bar{\mathbf{X}}$, the expected value is calculated by summing over all the proposals:

$$\begin{aligned} E\left[\frac{g(\bar{X}_1)}{q^*(\bar{X}_1)}|\bar{\mathbf{X}}\right] &= \sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{q^*(\bar{X}_{s,i})}P_r(\bar{X}_{s,i}|\bar{\mathbf{X}}) = \sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{\cancel{q^*(\bar{X}_{s,i})}}\cdot\frac{\cancel{q^*(\bar{X}_{s,i})}/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{i=1}^{M}q^*(\bar{X}_{s,i})/p(\bar{X}_{s,i})} \\ &= \sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-1}. \end{aligned} \tag{9}$$

We substitute these equations into Eq. (7) as:

$$
\begin{aligned}
E[\hat{I}_t|\bar{Z}_t] &= E\left[\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)\cdot\cancel{\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-1}}\cdot\frac{1}{M}\cancel{\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)}|\bar{Z}_t\right] \\
&= E\left[\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}|\bar{Z}_t\right] = \sum_{s\geq 1}\frac{1}{M}\sum_{i=1}^{M}E\left[\frac{g(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}|\bar{Z}_t\right] = \sum_{s\geq 1}\frac{1}{M}\sum_{i=1}^{M}E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right] = \sum_{s\geq 1}E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right].
\end{aligned}
$$

In summary, the expected value of the resampling estimator $\hat{I}_t$ is expressed as:

$$
\boxed{E[\hat{I}_t|\bar{Z}_t] = \sum_{s\geq 1}E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right].} \tag{10}
$$

We also prove that the resampling estimator is unbiased as:

$$
\boxed{E\left[\hat{I}_t|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}\sum_{s\geq 1}E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}\sum_{s\geq 1}\int_{A^s}g(\bar{y}\bar{Z}_t)d\mu(\bar{y}) = \frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}}g(\bar{y}\bar{Z}_t)d\mu(\bar{y}).} \tag{11}
$$

1.2 Derivation of $E[\hat{I}_t^2|\bar{Z}_t]$

$E[\hat{I}_t^2|\bar{Z}_t]$ can be calculated in the similar way as $E[\hat{I}_t|\bar{Z}_t]$. By using Eq. (7), $E[\hat{I}_t^2|\bar{Z}_t]$ is calculated by

$$
E\left[E\left[\left(\frac{1}{N}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{q^*(\bar{X}_j)}\right)^2|\bar{\mathbf{X}}\right]\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^2|\bar{Z}_t\right]. \tag{12}
$$

The expected value of the square of the sum is simplified as the following equation.

$$
E\left[\left(\frac{1}{N}\sum_{i=1}^{N}f(X_i)\right)^2\right] = \frac{1}{N}E[f(X_1)^2] + \left(1-\frac{1}{N}\right)E[f(X_1)]^2.
$$

By using this, the expected value of the square of the sum is calculated by:

$$
E\left[\left(\frac{1}{N}\sum_{j=1}^{N}\frac{g(\bar{X}_j)}{q^*(\bar{X}_j)}\right)^2|\bar{\mathbf{X}}\right] = \frac{1}{N}E\left[\left(\frac{g(\bar{X}_1)}{q^*(\bar{X}_1)}\right)^2|\bar{\mathbf{X}}\right] + \left(1-\frac{1}{N}\right)E\left[\frac{g(\bar{X}_1)}{q^*(\bar{X}_1)}|\bar{\mathbf{X}}\right]^2.
$$

Similar to Eq. (9), the expected value is calculated by summing over all the proposals as:

$$
\begin{aligned}
\frac{1}{N}E\left[\left(\frac{g(\bar{X}_1)}{q^*(\bar{X}_1)}\right)^2|\bar{\mathbf{X}}\right] &= \frac{1}{N}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})^2}{q^*(\bar{X}_{s,i})^2}\cdot\frac{q^*(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{i=1}^{M}q^*(\bar{X}_{s,i})/p(\bar{X}_{s,i})} \\
&= \frac{1}{N}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})^2}{q^*(\bar{X}_{s,i})p(\bar{X}_{s,i})}\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-1},
\end{aligned} \tag{13}
$$

$$\left(1-\frac{1}{N}\right)E\left[\frac{g(\bar{X}_1)}{q^*(\bar{X}_1)}|\bar{\mathbf{X}}\right]^2 = \left(1-\frac{1}{N}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{q^*(\bar{X}_{s,i})}\cdot\frac{q^*(\bar{X}_{s,i})/p(\bar{X}_{s,i})}{\sum_{s\geq 1}\sum_{i=1}^{M}q^*(\bar{X}_{s,i})/p(\bar{X}_{s,i})}\right)^2$$
$$= \left(1-\frac{1}{N}\right)\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^2\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^{-2}. \tag{14}$$

By substituting these equations into Eq. (12), the conditional variance $E[\hat{I}_t^2|\bar{Z}_t]$ is expressed by the following equation:

$$E[\hat{I}_t^2|\bar{Z}_t] = E\left[\frac{1}{N}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})^2}{q^*(\bar{X}_{s,i})p(\bar{X}_{s,i})}\cdot\frac{1}{M^2}\left(\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right] + E\left[\left(1-\frac{1}{N}\right)\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)^2|\bar{Z}_t\right]. \tag{15}$$

We first calculate the former expected value:

$$E\left[\frac{1}{N}\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{g(\bar{X}_{s,i})^2}{q^*(\bar{X}_{s,i})p(\bar{X}_{s,i})}\right)\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}\frac{q^*(\bar{X}_{s,i})}{p(\bar{X}_{s,i})}\right)|\bar{Z}_t\right]. \tag{16}$$

The expected value of the product of two sums is simplified to the following equation:

$$E\left[\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}f(X_{s,i})\right)\left(\frac{1}{M}\sum_{s\geq 1}\sum_{i=1}^{M}g(X_{s,i})\right)\right]$$
$$= \frac{1}{M}\sum_{s\geq 1}E[f(X_{s,1})g(X_{s,1})] - \frac{1}{M}\sum_{s\geq 1}E[f(X_{s,1})]E[g(X_{s,1})] + \left(\sum_{s\geq 1}E[f(X_{s,1})]\right)\left(\sum_{s\geq 1}E[g(X_{s,1})]\right). \tag{17}$$

This leads to the following equation:

$$\frac{1}{MN}\sum_{s\geq 1}E\left[\frac{g^2}{p^2}|\bar{Z}_t\right] - \frac{1}{MN}\sum_{s\geq 1}E\left[\frac{g^2}{pq^*}|\bar{Z}_t\right]E\left[\frac{q^*}{p}|\bar{Z}_t\right] + \frac{1}{N}\left(\sum_{s\geq 1}E\left[\frac{g^2}{pq^*}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1}E\left[\frac{q^*}{p}|\bar{Z}_t\right]\right), \tag{18}$$

where we omit the augment $X_{s,1}$ to simplify the notation.

Then the latter expected value in Eq. (15) is calculated by:

$$\frac{1}{M}\left(1-\frac{1}{N}\right)\sum_{s\geq 1}E\left[\frac{g^2}{p^2}|\bar{Z}_t\right] - \frac{1}{M}\left(1-\frac{1}{N}\right)\sum_{s\geq 1}\left(E\left[\frac{g}{p}|\bar{Z}_t\right]\right)^2 + \left(1-\frac{1}{N}\right)\left(\sum_{s\geq 1}E\left[\frac{g}{p}|\bar{Z}_t\right]\right)^2. \tag{19}$$

By summing Eqs. (18) and (19), $E[\hat{I}_t^2|\bar{Z}_t]$ is calculated by:

$$E[\hat{I}_t^2|\bar{Z}_t] = \frac{1}{M}\sum_{s\geq 1}E\left[\frac{g^2}{p^2}|\bar{Z}_t\right] - \frac{1}{MN}\sum_{s\geq 1}E\left[\frac{g^2}{pq^*}|\bar{Z}_t\right]E\left[\frac{q^*}{p}|\bar{Z}_t\right] + \frac{1}{N}\left(\sum_{s\geq 1}E\left[\frac{g^2}{pq^*}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1}E\left[\frac{q^*}{p}|\bar{Z}_t\right]\right)$$
$$-\frac{1}{M}\sum_{s\geq 1}\left(E\left[\frac{g}{p}|\bar{Z}_t\right]\right)^2 + \frac{1}{MN}\sum_{s\geq 1}\left(E\left[\frac{g}{p}|\bar{Z}_t\right]\right)^2 + \left(1-\frac{1}{N}\right)\left(\sum_{s\geq 1}E\left[\frac{g}{p}|\bar{Z}_t\right]\right)^2. \tag{20}$$

1.3 Derivation of $V[\hat{I}_t|\bar{Z}_t]$

The conditional variance $V[\hat{I}_t|\bar{Z}_t]$ is calculated by subtracting $E[\hat{I}_t|\bar{Z}_t]^2$ from $E[\hat{I}_t^2|\bar{Z}_t]$. By subtracting $E[\hat{I}_t|\bar{Z}_t]^2$ in Eq. (10) from Eq. (20) and rearranging the orders for better clarity, $V[\hat{I}_t|\bar{Z}_t]$ is calculated by:

$$V[\hat{I}_t|\bar{Z}_t] = \frac{1}{M}\left(\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})^2}{p(\bar{X}_{s,1})^2}|\bar{Z}_t\right] - \sum_{s\geq 1}\left(E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)^2\right) \tag{21}$$

$$- \frac{1}{MN}\left(\underbrace{\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})^2}{p(\bar{X}_{s,1})q^*(\bar{X}_{s,1})}|\bar{Z}_t\right] E\left[\frac{q^*(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]}_{E_a} - \underbrace{\sum_{s\geq 1}\left(E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)^2}_{E_b}\right) \tag{22}$$

$$+ \frac{1}{N}\left(\underbrace{\left(\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})^2}{p(\bar{X}_{s,1})q^*(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1} E\left[\frac{q^*(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)}_{E_c} - \underbrace{\left(\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)^2}_{E_d}\right). \tag{23}$$

In the following, we further simplify E_a, E_b, E_c, and E_d by using the following relations.

$$E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}\int_{A^s} g(\bar{y}\bar{Z}_t)d\mu(\bar{y}),$$

$$E\left[\frac{g(\bar{X}_{s,1})^2}{p(\bar{X}_{s,1})q^*(\bar{X}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}E\left[\frac{g(\bar{X}_{s,1})^2}{p(\bar{Y}_{s,1})q^*(\bar{X}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}\int_{A^s}\frac{g(\bar{y}\bar{Z}_t)^2}{q^*(\bar{y}\bar{Z}_t)}d\mu(\bar{y}), \tag{24}$$

$$E\left[\frac{q^*(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}E\left[\frac{q^*(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}|\bar{Z}_t\right] = \frac{1}{p(\bar{Z}_t)}\int_{A^s} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y}). \tag{25}$$

Substituting Eqs. (24) and (25) into E_a leads to

$$E_a = \frac{1}{p(\bar{Z}_t)^2}\sum_{s\geq 1}\left(\int_{A^s}\frac{g(\bar{y}\bar{Z}_t)^2}{q^*(\bar{y}\bar{Z}_t)}d\mu(\bar{y})\right)\left(\int_{A^s} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right) = \frac{1}{p(\bar{Z}_t)^2}\sum_{s\geq 1}\int_{A^s}\frac{g(\bar{y}\bar{Z}_t)^2}{q^*(\bar{y}\bar{Z}_t)/\int_{A^s} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y})}d\mu(\bar{y}).$$

The denominator of the integrand is represented by the conditional pdf $q_s(\bar{y}|\bar{Z}_t)$ as:

$$\frac{q^*(\bar{x}_s)}{\int_{A^s} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y})} = \frac{q^*(\bar{y},\bar{Z}_t)}{\int_{A^s} q^*(\bar{y},\bar{Z}_t)d\mu(\bar{y})} = q_s(\bar{y}|\bar{Z}_t). \tag{26}$$

Then E_a is expressed by the expected value with the pdf q_s as:

$$E_a = \frac{1}{p(\bar{Z}_t)^2}\sum_{s\geq 1}\int_{A^s}\frac{g(\bar{y}\bar{Z}_t)^2}{q_s(\bar{y}|\bar{Z}_t)}d\mu(\bar{y}) = \frac{1}{p(\bar{Z}_t)^2}\sum_{s\geq 1}E\left[\frac{g(\bar{X}_s)^2}{q_s(\bar{Y}_s|\bar{Z}_t)^2}|\bar{Z}_t\right] = \sum_{s\geq 1}E\left[\frac{g(\bar{X}_s)^2}{q_s(\bar{X}_s)^2}|\bar{Z}_t\right],$$

where X_s is a full path whose light sub-path $\bar{Y}_s$ follows the conditional pdf q_s, and we define the pdf $q_s(\bar{x}) = q_s(\bar{y}\bar{z}) = q_s(\bar{y}|\bar{z})p(\bar{z})$.

We change the pdf p in E_b into the pdf q_s as:

$$E_b = \sum_{s\geq 1}\left(\frac{1}{p(\bar{Z}_t)}E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)^2 = \sum_{s\geq 1}\left(\frac{1}{p(\bar{Z}_t)}\int_{A^s} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right)^2 = \sum_{s\geq 1}\left(E\left[\frac{g(\bar{X}_s)}{q_s(\bar{X}_s)}|\bar{Z}_t\right]\right)^2. \tag{27}$$

E_c is simplified in the same way as:

$$E_c = \left(\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})^2}{p(\bar{X}_{s,1})q^*(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)\left(\sum_{s\geq 1} E\left[\frac{q^*(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]\right) = \frac{1}{p(\bar{Z}_t)^2}\left(\sum_{s\geq 1}\int_{A^s}\frac{g^2(\bar{y}\bar{Z}_t)}{q^*(\bar{y}\bar{Z}_t)}d\mu(\bar{y})\right)\left(\sum_{s\geq 1}\int_{A^s} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right)$$

$$= \frac{1}{p(\bar{Z}_t)^2}\left(\int_{\mathcal{A}}\frac{g^2(\bar{y}\bar{Z}_t)}{q^*(\bar{y}\bar{Z}_t)}d\mu(\bar{y})\right)\left(\int_{\mathcal{A}} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right) = \frac{1}{p(\bar{Z}_t)^2}\int_{\mathcal{A}}\frac{g(\bar{y}\bar{Z}_t)^2}{q^*(\bar{y}\bar{Z}_t)/\int_{\mathcal{A}} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y})}d\mu(\bar{y}).$$

Similar to the conditional pdf $q_s(\bar{y}|\bar{z})$, we can represent the denominator of the integrand with the conditional pdf q as:

$$\frac{q^*(\bar{y}\bar{Z}_t)}{\int_{\mathcal{A}} q^*(\bar{y}\bar{Z}_t)d\mu(\bar{y})} = \frac{q^*(\bar{y},\bar{Z}_t)}{\int_{\mathcal{A}} q^*(\bar{y},\bar{Z}_t)d\mu(\bar{y})} = q(\bar{y}|\bar{Z}_t).$$

E_c is then expressed by the expected value with the pdf q as:

$$E_c = \frac{1}{p(\bar{Z}_t)^2}\int_{\mathcal{A}}\frac{g(\bar{y}\bar{Z}_t)^2}{q(\bar{y}|\bar{Z}_t)}d\mu(\bar{y}) = \frac{1}{p(\bar{Z}_t)^2}E\left[\frac{g(\bar{X})^2}{q(\bar{Y}|\bar{Z}_t)^2}|\bar{Z}_t\right] = E\left[\frac{g(\bar{X})^2}{q(\bar{X})^2}|\bar{Z}_t\right],$$

where $\bar{X} = \bar{Y}\bar{Z}_t$ is a full path whose light sub-path $\bar{Y}$ follows the conditional pdf $q(\bar{Y}|\bar{Z}_t)$, and we define the pdf $q(\bar{x}) = q(\bar{y}|\bar{z})p(\bar{z})$.

Finally, E_d is expressed as the expected value with the pdf q as:

$$E_d = \left(\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right]\right)^2 = \left(\frac{1}{p(\bar{Z}_t)}\sum_{s\geq 1}\int_{A^s} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right)^2 = \left(\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right)^2 = \left(E\left[\frac{g(\bar{X})}{q(\bar{X})}|\bar{Z}_t\right]\right)^2.$$

Substituting these equations into $V[\hat{I}_t|\bar{Z}_t]$ leads to the following equation:

$$\frac{1}{M}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right] - \frac{1}{MN}\left(\underbrace{\sum_{s\geq 1} E\left[\frac{g^2(\bar{X}_s)}{q_s^2(\bar{X}_s)}|\bar{Z}_t\right]}_{E_a} - \underbrace{\sum_{s\geq 1}\left(E\left[\frac{g(\bar{X}_s)}{q_s(\bar{X}_s)}|\bar{Z}_t\right]\right)^2}_{E_b}\right) + \frac{1}{N}\left(\underbrace{E\left[\frac{g(\bar{X})^2}{q(\bar{X})^2}|\bar{Z}_t\right]}_{E_c} - \underbrace{\left(E\left[\frac{g(\bar{X})}{q(\bar{X})}|\bar{Z}_t\right]\right)^2}_{E_d}\right).$$

In summary, $V[\hat{I}_t|\bar{Z}_t]$ (Eq. (8) in the paper) is expressed as:

$$\boxed{V[\hat{I}_t|\bar{Z}_t] = \frac{1}{M}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}|\bar{Z}_t\right] - \frac{1}{MN}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_s)}{q_s(\bar{X}_s)}|\bar{Z}_t\right] + \frac{1}{N}V\left[\frac{g(\bar{X})}{q(\bar{X})}|\bar{Z}_t\right].} \tag{28}$$

2 DERIVATION OF $V[\hat{I}_t]$

We now derive the variance $V[\hat{I}_t]$ by taking into accout the randomness of eye sub-path $\bar{Z}_t$. The variance $V[\hat{I}_t]$ is calculated by the law of total variance as:

$$V[\hat{I}_t] = E[V[\hat{I}_t|\bar{Z}_t]] + V[E[\hat{I}_t|\bar{Z}_t]]. \tag{29}$$

In the previous section, we have included the density $p(\bar{Z}_t)$ in the pdfs p, q_s, and q as constant. To take into account the randomness of $\bar{Z}_t$, we first decompose the pdfs p, q, and q_s into the pdf for the eye sub-path $p(\bar{Z}_t)$ and pdfs for light sub-paths as:

$$p(\bar{X}_{s,1}) = p(\bar{Y}_{s,1})p(\bar{Z}_t), q(\bar{X}) = q(\bar{Y}|\bar{Z}_t)p(\bar{Z}_t), q_s(\bar{X}_s) = q_s(\bar{Y}_s|\bar{Z}_t)p(\bar{Z}_t), \tag{30}$$

where q and q_s are conditional pdfs since $\bar{Y}$ and $\bar{Y}_s$ are resampled taking into account the eye sub-path $\bar{Z}_t$, while the pdf for light sub-path $\bar{Y}_{s,1}$ is independent of $\bar{Z}_t$ as the traditional BPT.

By substituting Eq. (30) into Eqs. (10) and (28), the conditional expected value $E[\hat{I}_t|\bar{Z}_t]$ and the conditional variance $V[\hat{I}_t|\bar{Z}_t]$ is expressed as:

$$E[\hat{I}_t|\bar{Z}_t] = \frac{1}{p(\bar{Z}_t)}\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}\right],$$

$$V[\hat{I}_t|\bar{Z}_t] = \frac{1}{p(\bar{Z}_t)^2}\left(\frac{1}{M}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}|\bar{Z}_t\right] - \frac{1}{MN}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_s)}{q_s(\bar{Y}_s|\bar{Z}_t)}|\bar{Z}_t\right] + \frac{1}{N}V\left[\frac{g(\bar{X})}{q(\bar{Y}|\bar{Z}_t)}|\bar{Z}_t\right]\right).$$

By substituting above two equations into Eq. (29), $V[\hat{I}_t]$ is expressed by:

$$\underbrace{\frac{1}{M}\sum_{s\geq 1} E\left[\frac{1}{p(\bar{Z}_t)^2}V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}|\bar{Z}_t\right]\right] - \frac{1}{MN}\sum_{s\geq 1} E\left[\frac{1}{p(\bar{Z}_t)^2}V\left[\frac{g(\bar{X}_s)}{q_s(\bar{Y}_s|\bar{Z}_t)}|\bar{Z}_t\right]\right] + \frac{1}{N}E\left[\frac{1}{p(\bar{Z}_t)^2}V\left[\frac{g(\bar{X})}{q(\bar{Y}|\bar{Z}_t)}|\bar{Z}_t\right]\right]}_{E[V[\hat{I}_t|\bar{Z}_t]]}$$

$$\underbrace{+\,V\left[\frac{1}{p(\bar{Z}_t)}\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}\right]\right]}_{V[E[\hat{I}_t|\bar{Z}_t]]}.$$

By using $E[V[a|b]b^2] = V[ab] - V[E[a|b]b]$ and considering $\frac{1}{p(\bar{Z}_t)}$ as a random variable of $\bar{Z}_t$,

$$\begin{aligned}
\frac{1}{M}\sum_{s\geq 1} E\left[\frac{1}{p(\bar{Z}_t)^2}V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}|\bar{Z}_t\right]\right] &= \frac{1}{M}\sum_{s\geq 1}\left(V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}\cdot\frac{1}{p(\bar{Z}_t)}\right] - V\left[\frac{1}{p(\bar{Z}_t)}E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}|\bar{Z}_t\right]\right]\right)\\
&= \frac{1}{M}\sum_{s\geq 1}\left(V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}\right] - V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right]\right),\\
\frac{1}{MN}\sum_{s\geq 1} E\left[\frac{1}{p(\bar{Z}_t)^2}V\left[\frac{g(\bar{X}_s)}{q_s(\bar{Y}_s|\bar{Z}_t)}|\bar{Z}_t\right]\right] &= \frac{1}{MN}\sum_{s\geq 1}\left(V\left[\frac{g(\bar{X}_s)}{q_s(\bar{Y}_s|\bar{Z}_t)}\cdot\frac{1}{p(\bar{Z}_t)}\right] - V\left[\frac{1}{p(\bar{Z}_t)}E\left[\frac{g(\bar{X}_s)}{q_s(\bar{Y}_s|\bar{Z}_t)}|\bar{Z}_t\right]\right]\right)\\
&= \frac{1}{MN}\sum_{s\geq 1}\left(V\left[\frac{g(\bar{X}_s)}{q_s(\bar{X}_s)}\right] - V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right]\right),\\
\frac{1}{N}E\left[\frac{1}{p(\bar{Z}_t)^2}V\left[\frac{g(\bar{X})}{q(\bar{Y}|\bar{Z}_t)}|\bar{Z}_t\right]\right] &= \frac{1}{N}\left(V\left[\frac{g(\bar{X})}{q(\bar{Y}|\bar{Z}_t)}\cdot\frac{1}{p(\bar{Z}_t)}\right] - V\left[\frac{1}{p(\bar{Z}_t)}E\left[\frac{g(\bar{X})}{q(\bar{Y}|\bar{Z}_t)}|\bar{Z}_t\right]\right]\right)\\
&= \frac{1}{N}\left(V\left[\frac{g(\bar{X})}{q(\bar{X})}\right] - V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right]\right),\\
V\left[\frac{1}{p(\bar{Z}_t)}\sum_{s\geq 1} E\left[\frac{g(\bar{X}_{s,1})}{p(\bar{Y}_{s,1})}\right]\right] &= V\left[\frac{1}{p(\bar{Z}_t)}\sum_{s\geq 1}\int_{A^s} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right] = V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right].
\end{aligned}$$

By summing up all the above equations, the variance $V[\hat{I}_t]$ (Eq. (9) in the paper) is expressed by:

$$V[\hat{I}_t] = \frac{1}{M}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}\right] - \frac{1}{MN}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_s)}{q_s(\bar{X}_s)}\right] + \frac{1}{N}V\left[\frac{g(\bar{X})}{q(\bar{X})}\right]$$
$$- \frac{1}{M}\left(1-\frac{1}{N}\right)\sum_{s\geq 1} V\left[\frac{1}{p(\bar{Z}_t)}\int_{A^s} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right] + \left(1-\frac{1}{N}\right)V\left[\frac{1}{p(\bar{Z}_t)}\int_{\mathcal{A}} g(\bar{y}\bar{Z}_t)d\mu(\bar{y})\right]. \tag{31}$$

To make the minimization problem of the variance $V[\hat{I}_t]$ feasible, we set the number of resampling light sub-path samples N to one. This simplifies the above equation to the following (Eq. (10) in the paper):

$$V[\hat{I}_t] = \frac{1}{M}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_{s,1})}{p(\bar{X}_{s,1})}\right] - \frac{1}{M}\sum_{s\geq 1} V\left[\frac{g(\bar{X}_s)}{q_s(\bar{X}_s)}\right] + V\left[\frac{g(\bar{X})}{q(\bar{X})}\right]. \tag{32}$$

3 DERIVATION OF WEIGHTING FUNCTIONS w_t

We derive the weighting functions to reduce the variance $V[\hat{I}]$ of the pixel measurement I. We first define the variance $V[\hat{I}]$ and then derive the weighting function.

3.1 Definition of the variance $V[\hat{I}]$

The pixel measurement I is estimated by a sum of estimators with different sampling techniques as:

$$\hat{I} = \sum_{t\geq 2}\hat{I}_t + \sum_{s\geq 2}\hat{I}_{s,0} + \sum_{t\geq 2}\hat{I}_{0,t} + \sum_{s\geq 1}\hat{I}_{s,1}, \tag{33}$$

where $\hat{I}_{s,t}$ is the estimate of the contributions from the paths sampled by connecting the t-th vertex of an eye sub-path $\bar{z}$ and the s-th vertex of a light sub-path $\bar{y}$. $\hat{I}_{s,0}$ and $\hat{I}_{0,t}$ estimate the contributions of the paths sampled by using unidirectional sampling from light sources and the camera, respectively. $\hat{I}_{s,1}$ estimates the contributions of the paths sampled by using light tracing.

The variance $V[\hat{I}]$ is calculated by:

$$V[\hat{I}] = \sum_{t\geq 2} V[\hat{I}_t] + \sum_{s\geq 2}\frac{1}{N_{s,0}}V[\hat{I}_{s,0}] + \sum_{t\geq 2}\frac{1}{N_{0,t}}V[\hat{I}_{0,t}] + \sum_{s\geq 1}\frac{1}{N_{s,1}}V[\hat{I}_{s,1}]. \tag{34}$$

where $N_{s,0}$, $N_{0,t}$, and $N_{s,1}$ are the number of samples for each sampling technique, and the number of samples for the resampling estimator $\hat{I}_t$ (i.e., N) is one (and thus the reciprocal of N is omitted).

We now consider a path $\bar{x} = x_0 \dots x_k$ with length k, and $k+2$ strategies can take the same path $\bar{x}$. These $k+2$ strategies can be identified by the number of the eye sub-path vertices, t, and the number of the light sub-path vertices is specified uniquely as $s = k - t + 1$. $k-1$ strategies ($t = 2, \dots, k$) are handled by our method. The other sampling strategies, namely unidirectional sampling ($s = 0, t \geq 2$), ($s \geq 2, t = 0$) and light tracing ($s \geq 1, t = 1$) for a path with length k are also represented by $t = k+1$, $t = 0$, and $t = 1$, respectively. These three strategies ($t = 0, 1, k+1$) are handled by BPT. We define $\Lambda_{IS} = \{0, 1, k+1\}$ and $\Lambda_{RIS} = \{2, \dots, k\}$ to represent these two types of strategy.

For a path $\bar{x}$ with length k, the variance $V[\hat{I}]$ in Eq. (34) is rewritten as:

$$V[\hat{I}] = \sum_{t\in\Lambda_{RIS}} V[\hat{I}_t] + \frac{1}{N_{k+1,0}}V[\hat{I}_{k+1,0}] + \frac{1}{N_{0,k+1}}V[\hat{I}_{0,k+1}] + \frac{1}{N_{k,1}}V[\hat{I}_{k,1}]$$

To simplify the notation, we gather the last three terms handled by BPT using Λ_{IS} and represent the number of samples $N_{s,t}$ with N_t. By substituting Eq. (32) for $V[\hat{I}_t]$, the variance is rewritten as:

$$V[\hat{I}] = \sum_{t\in\Lambda_{RIS}} \left(\frac{1}{M}V\left[\frac{w_t f}{p}\right] - \frac{1}{M}V\left[\frac{w_t f}{q_s}\right] + V\left[\frac{w_t f}{q}\right]\right) + \sum_{t\in\Lambda_{IS}} \frac{1}{N_t}V\left[\frac{w_t f}{p}\right], \tag{35}$$

where g is replaced with $w_t f$, N_t is the number of samples handled by BPT, and the arguments are omitted for simplicity. The summation over s in Eq. (32) is eliminated since the number of the light sub-path vertices is uniquely determined as $s = k - t + 1$.

3.2 Derivation of weighting functions

We derive the weighting functions in the same sense as the balance heuristic. That is, we base on the following premises that are used in the derivation of the balance heuristic [Veach 1997] to reduce the variance $V[\hat{I}]$.

- Minimize $E[\hat{I}^2]$ as the upper bound of the variance $V[\hat{I}] = E[\hat{I}^2] - E[\hat{I}]^2$, instead of minimizing the variance $V[\hat{I}]$ itself.
- Do not consider the covariance terms (i.e., the correlations of light sub-paths).

Instead of minimizing the variance V in Eq. (35) itself, we minimize the second moment $E[\hat{I}^2]$ as the upper bound in the same way as the balance heuristic:

$$\begin{aligned} E[\hat{I}^2] &= \sum_{t\in\Lambda_{RIS}} \left(\frac{1}{M}E\left[\frac{w_t^2 f^2}{p^2}\right] - \frac{1}{M}E\left[\frac{w_t^2 f^2}{q_s^2}\right] + E\left[\frac{w_t^2 f^2}{q^2}\right]\right) + \sum_{t\in\Lambda_{IS}} \frac{1}{N_t}E\left[\frac{w_t^2 f^2}{p^2}\right] \\ &= \sum_{t\in\Lambda_{RIS}} \left(\frac{1}{M}\int \frac{w_t^2(\bar{x})f^2(\bar{x})}{p(\bar{x})}d\mu(\bar{x}) - \frac{1}{M}\int \frac{w_t^2(\bar{x})f^2(\bar{x})}{q_s(\bar{x})}d\mu(\bar{x}) + \int \frac{w_t^2(\bar{x})f^2(\bar{x})}{q(\bar{x})}d\mu(\bar{x})\right) \\ &+ \sum_{t\in\Lambda_{IS}} \frac{1}{N_t}\int \frac{w_t^2(\bar{x})f^2(\bar{x})}{p(\bar{x})}d\mu(\bar{x}). \end{aligned} \tag{36}$$

Since it is sufficient to minimize the integrand at each path $\bar{x}$ separately and $f(\bar{x})$ is constant for all strategies, we minimize the following objective function:

$$\sum_{t\in\Lambda_{RIS}} \left(\frac{1}{M}\frac{w_t^2(\bar{x})}{p(\bar{x})} - \frac{1}{M}\frac{w_t^2(\bar{x})}{q_s(\bar{x})} + \frac{w_t^2(\bar{x})}{q(\bar{x})}\right) + \sum_{t\in\Lambda_{IS}} \frac{1}{N_t}\frac{w_t(\bar{x})^2}{p(\bar{x})}, \tag{37}$$

subject to the condition $\sum_{t=0}^{k+1} w_t = 1$. We further simplify the objective function by using the following relation between the pdfs q and q_s:

$$q_s(\bar{x}) = \frac{q^*(\bar{x})}{\int_{A^s} q^*(\bar{x})d\mu(\bar{y})} = \frac{q^*(\bar{x})}{\int_{\mathcal{A}} q^*(\bar{x})d\mu(\bar{y})} \cdot \frac{\int_{\mathcal{A}} q^*(\bar{x})d\mu(\bar{y})}{\int_{A^s} q^*(\bar{x})d\mu(\bar{y})} = q(\bar{x})\frac{Q}{Q_s}, \tag{38}$$

where Q and Q_s are normalization factors. By substituting this, the objective function is expressed by:

$$\sum_{t\in\Lambda_{RIS}} \left(\frac{1}{M}\frac{1}{p(\bar{x})} + \left(1 - \frac{1}{M}\frac{Q_s}{Q}\right)\frac{1}{q(\bar{x})}\right) w_t^2(\bar{x}) + \sum_{t\in\Lambda_{IS}} \frac{1}{N_t}\frac{w_t(\bar{x})^2}{p(\bar{x})}, \tag{39}$$

To simplify and unify the notations, we define the following function p_{ris}:

$$p_{\mathrm{ris}}(\bar{x}) = \left(\frac{1}{M}\frac{1}{p(\bar{x})} + \left(1 - \frac{1}{M}\frac{Q_s}{Q}\right)\frac{1}{q(\bar{x})}\right)^{-1}. \tag{40}$$

Then the objective function is simplified as:

$$\sum_{t=0}^{k+1} \frac{w_t^2(\bar{x})}{n_t p_t(\bar{x})}. \tag{41}$$

This enables the same derivation of the weighting functions of the balance heuristic [Veach 1997, p. 289] as:

$$w_t(\bar{x}) = \frac{n_t p_t(\bar{x})}{\sum_{i=0}^{k+1} n_i p_i(\bar{x})}, \tag{42}$$

$$n_t = \begin{cases} 1 & (t \in \Lambda_{RIS}) \\ N_t & (t \in \Lambda_{IS}) \end{cases}, \tag{43}$$

$$p_t(\bar{x}) = \begin{cases} \left(\frac{1}{M}\frac{1}{p(\bar{x})} + \left(1 - \frac{1}{M}\frac{Q_s}{Q}\right)\frac{1}{q(\bar{x})}\right)^{-1} & (t \in \Lambda_{RIS}) \\ p(\bar{x}) & (t \in \Lambda_{IS}). \end{cases} \tag{44}$$

4 ADDITIONAL RESULTS

San Miguel scene is a relatively vast scene illuminated by an environment map with high directionality, but the camera is only aimed at a small detail of this scene, whici is hard to deal with BPT. Fig. 1 shows the convergence graph of MAPEs for San Miguel scene. Although the noise reduction for $M = 400$ is unstable due to the effects of light sub-paths with high contribution, our method (solid lines) can converge consistently faster than BPT (dot-dashed line) and PCBPT (dashed lines).

Fig. 2 shows the iteration counts and MAPEs for various numbers of light sub-paths M and resampling light sub-paths N. Similar to the other scenes of Fig. 4 in the main paper, $M = 200$ and $N = 1$ works fine in San Miguel scene. The iteration counts decrease for larger M and N as expected. We evaluated the effects of the number of nearest cache points N_c for the San Miguel scene. MAPEs for $N_c = 1, 3, 5$ are 0.134, 0.121, and 0.129, respectively.

5 EQUAL-TIME COMPARISONS USING DIFFERENT TARGET DISTRIBUTIONS

Fig. 3 shows equal-time comparisons between BPT, PCBPT, and our method. We have used two target distributions for PCBPT and our method, $q^* = f_y f_{yz}$ and $q^* = f_y \rho GV$ (Eq. (15) in the paper), where f_y and f_{yz} are a part of the contribution function f that depends only on the light sub-path $\bar{y}$ and depends on both the light sub-path $\bar{y}$ and the eye sub-path $\bar{z}$, respectively. ρ and GV are the BSDF and the geometry term including the visibility, respectively. The latter target distribution $q^* = f_y \rho GV$ is equal to the former target distribution $q^* = f_y f_{yz}$ without the BSDF at the last vertex of the eye sub-path. Classroom, Sponza, and Door scenes are rendered in 1 min, while Bedroom, House, and SanMiguel scenes are rendered in 10 min. Beroom and House scenes are rendered using our method combined with path guiding and PPM, respectively. In PCBPT and our method, the numbers of pre-sampled light sub-paths M are set to $M = 100$ and $M = 200$, respectively. In all scenes, our method yields better performance compared to BPT and PCBPT, and our method consistently yields smaller MAPEs than PCBPT for both target distributions. This indicates that the performance gain in our method comes from our resampling-aware weihting function.

Fig. 4 shows the convergence plots for PCBPT and our method using two target distributions $q^* = f_y f_{yz}$ (dashed line) and $q^* = f_y \rho GV$ (Eq. (15) in the paper, solid line). In all scenes, our method converges faster than PCBPT for two target distributions. We have decided to use the latter target distribution $q^* = f_y \rho GV$ since it yields better results as shown in Figs. 3 and 4. Moreover, as pronounced in (f) San Miguel scene, unnatural undularion in the plot is shown (around 20s) using the target distribution $q^* = f_y f_{yz}$ (while it eventually converges faster than

PCBPT). The use of the target distribution $q^* = f_y f_{yz}$ that includes the BSDF at the last vertex of the eye sub-path makes the estimate of the normalization factor Q unstable (especially for a small number of pre-sampled light sub-paths M), resulting in unstable variance reduction as shown in Fig. 4 (f).

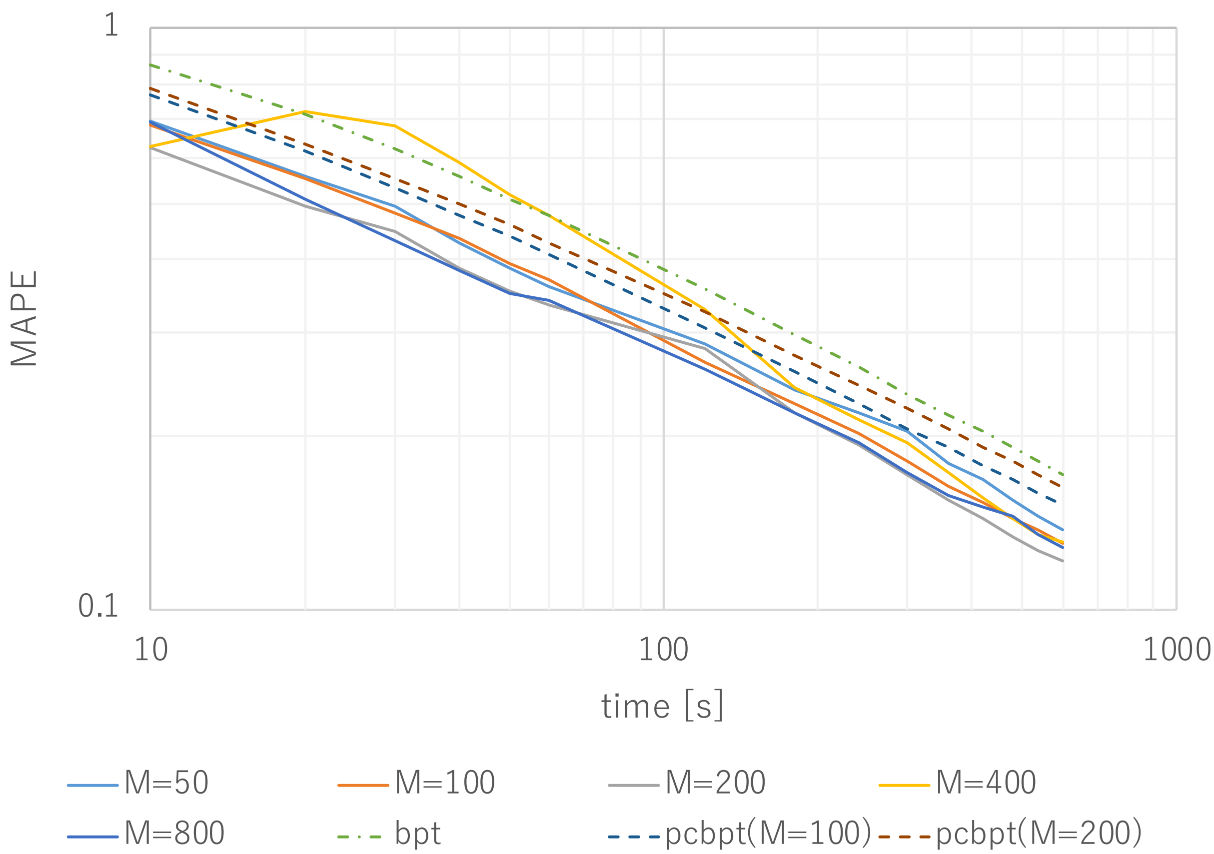


Fig. 1. Convergence graph of MAPEs for San Miguel scene with different numbers of light sub-paths, M. Our method (solid lines) with $M = 50, 100, 200, 400, 800$ converges consistently faster than BPT and PCBPT (dashed lines).

REFERENCES

Eric Veach. 1997. *Robust Monte Carlo Methods for Light Transport Simulation.* Ph.D. Dissertation. Stanford University.

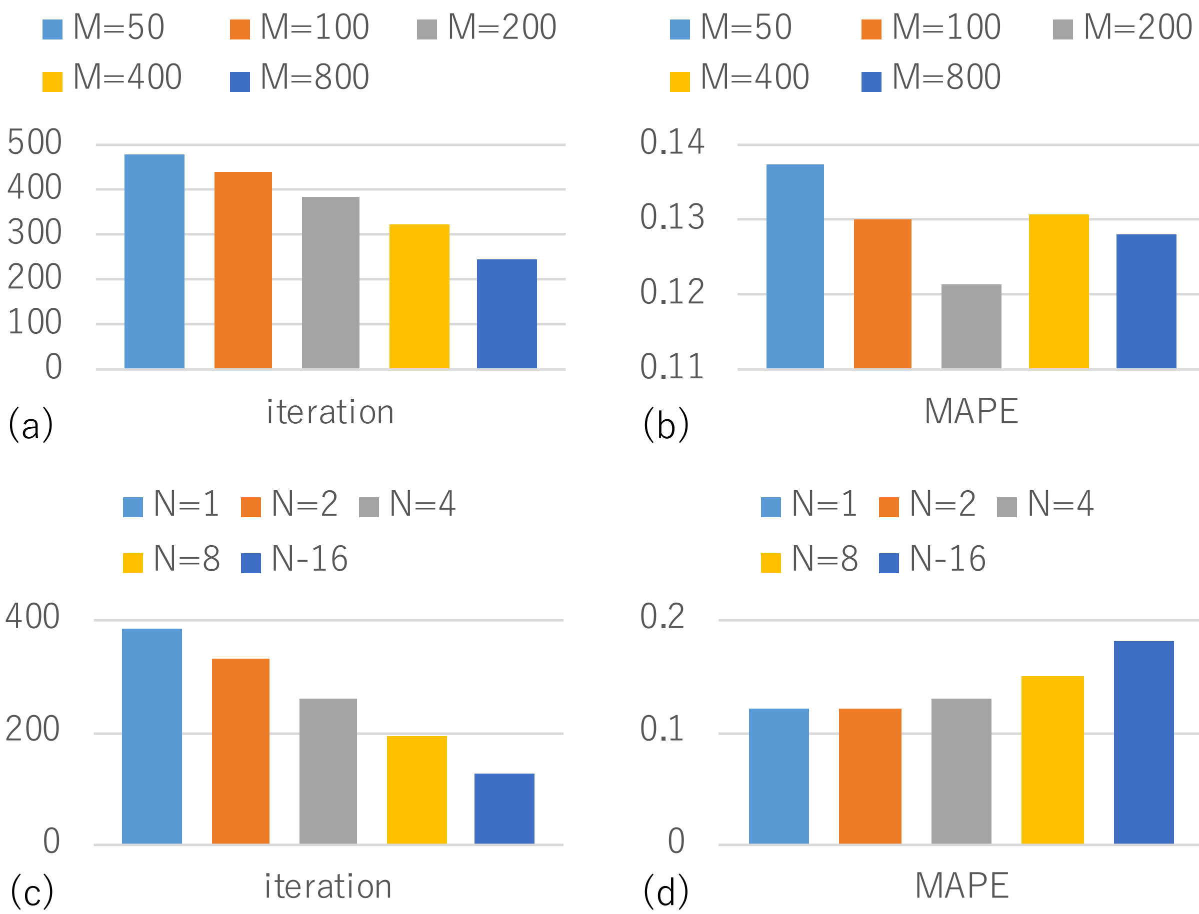
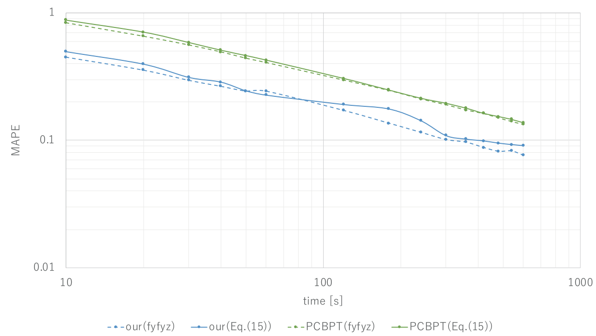
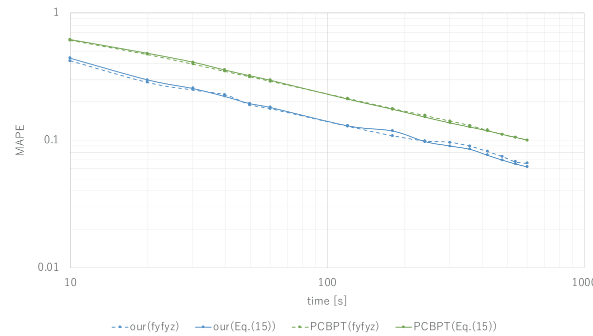


Fig. 2. Statistics for San Miguel scene. (a) iteration counts and (b) MAPEs for $M = 50, 100, 200, 400$, and 800. (c) iteration counts and (d) MAPEs for various numbers of resampling light sub-paths $N = 1, 2, 4, 8$, and 16. In (a) and (b), $N = 1$ is used, and in (c) and (d), $M = 200$ is used.

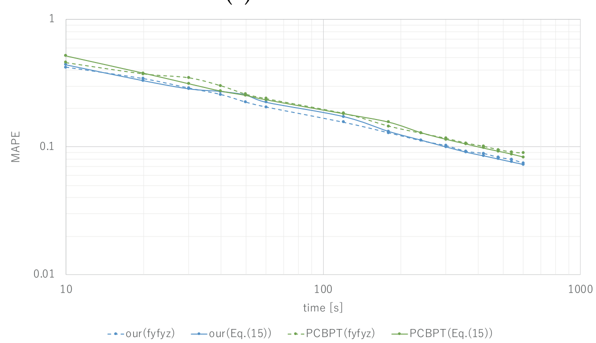
Fig. 3. Equal-time comparison between BPT, PCBPT ($M = 100$), and our method ($M = 200$) for Classroom, Sponza, Door, Bedroom, House, and SanMiguel scenes. For PCBPT and our method, we used two target distributions $q^*(\bar{x}) = f_y(\bar{y})\rho GV$ (Eq. (15) in the paper) and $q^*(\bar{x}) = f_y(\bar{y})f_{yz}(\bar{y}, \bar{z})$. The numbers shown in the bottom are MAPEs. As shown in the images and MAPEs, our method yields better results for both target distributions.



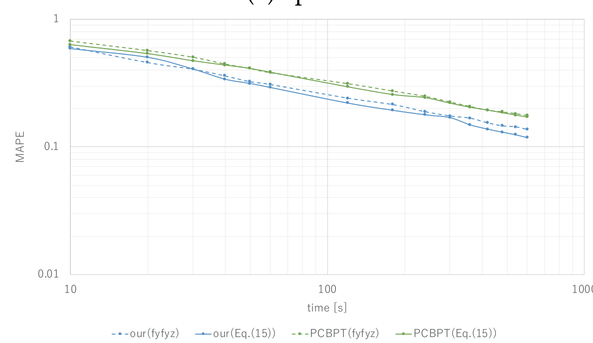
(a) Classroom



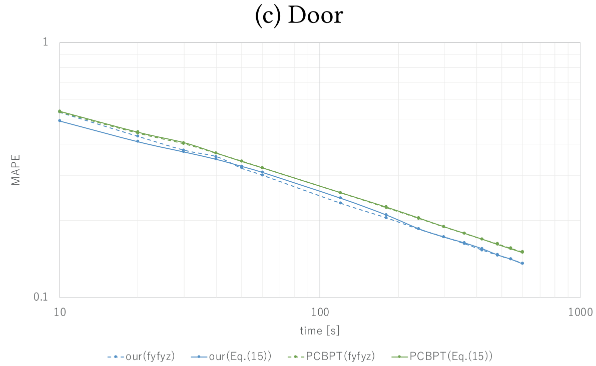
(b) Sponza



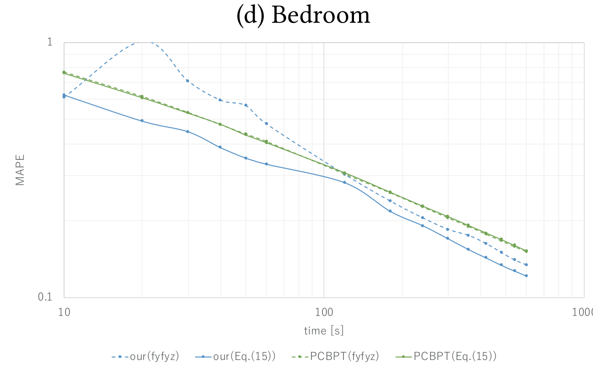
(c) Door



(d) Bedroom



(e) House



(f) SanMiguel

Fig. 4. Convergence plots of MAPEs for our method and PCBPT with different target distributions $q^* = f_y f_{yz}$ (dashed line) and $q^* = f_y \rho G V$ (solid line). Our method converges faster than PCBPT for both target distributions.